MATH203 Calculus

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Independent of path

Theorem 1

If $F(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is continuous on an open connected region D, then the line integral $\int_C \mathbf{F} \cdot dr$ is independent of path if and only if $\mathbf{F}$ is conservative that is $\mathbf{F}(x, y) = \nabla f(x, y)$ for some scalar function.

Theorem 2: Fundamental theorem of line integrals

Let $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ be continuous on an open connected region D, and let C be a piecewise smooth curve in D with endpoint $A(x_1, y_1)$ and $B(x_2, y_2)$. If $\mathbf{F}(x, y) = \nabla f(x, y)$, then

$$\int_C M(x, y)dx + N(x, y)dy = \int_{(x_1,y_1)}^{(x_2,y_2)} \mathbf{F} \cdot dr = \Big[f(x, y)\Big]_{(x_1,y_1)}^{(x_2,y_2)}$$

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Theorem 3

If $M(x,y)$ and $N(x,y)$ have continuous first partial derivatives on a simply connected region D, then the line integral $\int_C M(x,y)dx + N(x,y)dy$ is independent of path in D if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example 1: Let $\mathbf{F}(x,y) = (2x+y^3)\mathbf{i} + (3xy^2+4)\mathbf{j}$

(a) Show that $\int_C \mathbf{F} \cdot dr$ is independent of path.

(b) $\int_{(0,1)}^{(2,3)} \mathbf{F} \cdot dr$.

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Example 2: Show that $\int_C (e^{3y} + y^2 \sin x)dx + (3xe^{3y} + 2y \cos x)dy$ is independent of path in a simply connected region.

Example 3: Determine whether $\int_C x^2 y dx + 3xy^2 dy$ is independent of path.

Example 4: Let $\mathbf{F}(x, y, z) = y^2 \cos x\mathbf{i} + (2y \sin x + e^{2z})\mathbf{j} + 2ye^{2z}\mathbf{k}$

(a) Show that $\int_C \mathbf{F} \cdot dr$ is independent of path, and find a potential function f of $\mathbf{F}$.

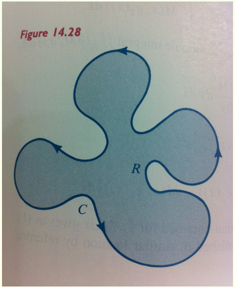
(b) If $\mathbf{F}$ is a force field, find work done by $\mathbf{F}$ along any curve C from $(0, 1, \frac{1}{2})$ to $(\frac{\pi}{2}, 3, 2)$.

Green's Theorem

Green's theorem

Let C be a piecewise-smooth simple closed curve, and let R be the region consisting of C and its interior. If M and N are continuous functions that have continuous first partial derivatives throughout an open region D containing R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$



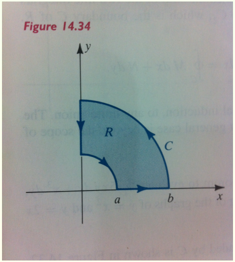
Green's Theorem

Note: Note the line integral is independent of path and hence is zero $\int_C \mathbf{F}\cdot dr = 0$ for every simple closed curve C.

Examples: (1) Use Green's theorem to evaluate $\oint_C 5xydx + x^3dy$, where C is the closed curve consisting of the graphs of $y = x^2$ and $y = 2x$ between the points $(0,0)$ and $(2,4)$.

(2) Use Green's theorem to evaluate $\oint_C 2xydx + (x^2+y^2)dy$, if C is the ellipse $4x^2 + 9y^2 = 36$.

(3) Evaluate $\oint_C (4 + e^{\cos x})dx + (\sin y + 3x^2)dy$, if C the boundary of the region R between quarter-circles of radius a and b and segment on the $x-$ and $y-$axes, as shown in Figure.



Green's Theorem

Theorem

If a region R in the $xy-$plane is bounded by a piecewise-smooth simple closed curve C, then the area A of R is

$$\begin{aligned}\iint\limits_R dA &= \oint_C x\,dy \qquad \text{(i)}\\ &= -\oint_C y\,dx \qquad \text{(ii)}\\ &= \tfrac{1}{2}\oint_C x\,dy - y\,dx. \qquad \text{(iii)}\end{aligned}$$

Examples: (1) Find the area of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$.

(2) Find the area of the region bounded by the graphs of $y = 4x^2$ and $y = 16x$.