MATH203 Calculus

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Alternating Series

Definition

The alternating Series

$$\sum_{n=1}^{\infty}(-1)^n a_n = -a_1 + a_2 + \cdots + (-1)^n a_n + \dots \text{ or}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1} a_n = a_1 - a_2 + \cdots + (-1)^{n-1} a_n + \dots$$

Alternating Series Test (AST)

If $a_n > 0$, then the alternating Series $\sum_{n=1}^{\infty}(-1)^n a_n$ or. $\sum_{n=1}^{\infty}(-1)^{n-1} a_n$ converge if the following conditions are satisfied.

1. $a_n \geqslant a_{n+1} > 0$.
2. $\lim_{n\to\infty} a_n = 0$.

If condition (2) in AST is not satisfied then the series is d'gt.

Examples

Determine whether the alternating series converges or diverges

(1): $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{4n^2-3}$ (2): $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{4n-3}$

(3): $\sum_{n=1}^{\infty}(-1)^{n-1}n5^{-n}$ (4): $\sum_{n=1}^{\infty}(-1)^{n}\frac{n+1}{n}$

Solution:

Absolute convergence

Definition

A series $\sum_{n=1}^{\infty} |a_n|$ is called an absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + \cdots + |a_n| + \ldots$ is convergent.

Conditionally convergent Series

Definition

A series $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if the series $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} |a_n|$ is divergent.

Theorem

If a series $\sum_{n=1}^{\infty} |a_n|$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is convergent (converse is not true).

Examples

Determine whether the series is absolute convergent, conditionally convergent or divergent

(1): $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$ (2): $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{1}{n}$

(3): $\sum_{n=1}^{\infty}(-1)^{n-1} \frac{1}{\sqrt{n}}$ (4): $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+1}$

Solution:

Conditionally convergent Series

Remark

For any series $\sum_{n=1}^{\infty}(-1)^n a_n$ exactly one of the following statements is true:

- It is absolutely convergent.
- It is conditionally convergent.
- It is divergent.

Absolute Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of non-zero terms, suppose $\sum_{n=1}^{\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$, then

- the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $L < 1$.
- the series $\sum_{n=1}^{\infty} a_n$ is divergent if $L > 1$ or $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$.
- If $L = 1$ (fails), the series may converge or diverge.

Examples

Determine whether the series is absolute convergent, conditionally convergent or divergent

(1): $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2+4}{2^n}$

(2): $\displaystyle\sum_{n=1}^{\infty}\frac{(-10)^n}{n!}$

(3): $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^4}{e^n}$

Solution:

Examples

Determine whether the series is absolute convergent, conditionally convergent or divergent

(1): $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2n+1}{3n+2}$

(2): $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{\sqrt{n+2}}$

(3): $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n+1}}$

(4): $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{5n-1}$

Solution:

Power Series

Definition

If x is a variable, then an intinite series of the form $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_2 x + \cdots + a_n x^n + \ldots;\ a_i \in \mathbb{R}$ is called a power series in x or. $\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_2(x-c) + \cdots + a_n (x-c)^n + \ldots;\ c \in \mathbb{R}$ is called a power series in $(x-c)$

Remarks:

1. We can check the convergence or divergence of a power series $\sum_{n=0}^{\infty} a_n x^n$ for different values of x.
2. Every power series in x converges if $x = 0$.
3. To find all other values of x for which $\sum_{n=0}^{\infty} a_n x^n$ is convergent, we often use **the absolute ratio test**.

Interval of convergence

After finding values of x which are convergent in the interval, say (a,b), this is called the interval of convergence for the power series $\sum_{n=0}^{\infty} a_n x^n$.

Radius of convergence

Half of the length of interval of convergence is called the radius of convergence of the the power series $\sum_{n=0}^{\infty} a_n x^n$.

Theorem

Every power series $\sum_{n=0}^{\infty} a_n x^n$ satisfies one of the following:

1. The series converges only when $x = 0$ and this convergence is absolute.
2. The series converges for all x, and this convergence is absolute.
3. There is a number $R >$ such that the series converges absolutely when $x < R$ and diverges when $x > R$. Note that the series may converge or diverge depending on the particular series.

Examples

Find the interval of convergence and radius of convergence of the following series:

(1): $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n} x^n$ (2): $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (3): $\displaystyle\sum_{n=0}^{\infty} (n!) x^n$

(4): $\displaystyle\sum_{n=0}^{\infty} (2x)^n \frac{1}{n}$ (5): $\displaystyle\sum_{n=0}^{\infty} x^n \frac{1}{\sqrt{n}}$ (6): $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} (x-3)^n$

Solution:

Examples

Determine whether the series is absolute convergent, conditionally convergent or divergent

(1): $\displaystyle\sum_{n=1}^{\infty}(-1)^n(1+e^{-n})$ (2): $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n}{\ln n}$

(3): $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2+3}{(2n-5)^2}$ (4): $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{(-5)^n}$

Solution:

Examples

Find the interval of convergence and radius of convergence of the following series:

(1): $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+4} x^n$ (2): $\displaystyle\sum_{n=0}^{\infty} \frac{x^n n^2}{2^n}$

(3): $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}} x^n$ (4): $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{10^n} (x-4)^n$

Solution: