MATH203 Calculus

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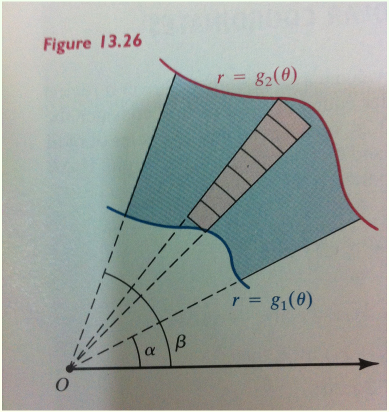
Double Integrals in polar coordinates



Double Integrals in polar coordinates

Evaluation theorem:

$$\lim_{\|P\|\to 0} \sum_k f(r_k, \theta_k) r_k \Delta r_k \Delta \theta_k = \iint_R f(r,\theta)\mathrm{d}A = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta) r \mathrm{d}r \mathrm{d}\theta \tag{1}$$

Note:

If $f(r,\theta) = 1$ throughout R, then the above double integral (1) equals the area of R

$$\int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} r \mathrm{d}r \mathrm{d}\theta$$

Double Integrals in polar coordinates

Some useful polar graphs

1- Straight line $y = 3$ has the polar equation

$$r = 3\csc\theta$$

2- Straight line $x = 2$ has the polar equation

$$r = 2\sec\theta$$

3- Circle $x^2 + y^2 = 4$ is $r = 2$.
4- Circle $(x-2)^2 + y^2 = 4$ is $r = 4\cos\theta$.
5- Circle $x^2 + (y-2)^2 = 4$ is $r = 4\sin\theta$.
6-Cardiods $r = a(1+\cos\theta)$ and $r = a(1+\sin\theta)$.

Double Integrals in polar coordinates

Examples

(1) Find the area of the region R that lies outside the circle $r = a$ and inside the circle $r = 2a\sin\theta$, $a > 0$.

(2) Find the area of the region R bounded by one loop of the lemniscate $r^2 = a^2 \sin 2\theta$ where $a > 0$.

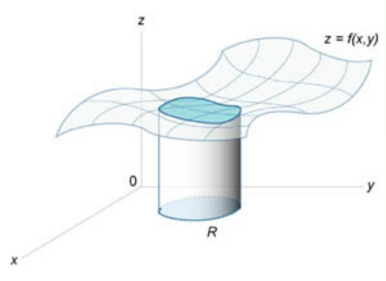
(3) Use polar coordinates to evaluate

a-

$$\int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} (x^2+y^2)^{3/2} \mathrm{d}y\mathrm{d}x.$$

b- $\displaystyle\iint_R (x+y)\mathrm{dA}$; R is bounded by the circle $x^2+y^2=2y$.

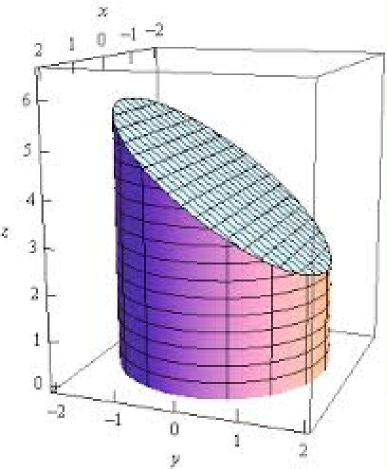
Surface Area



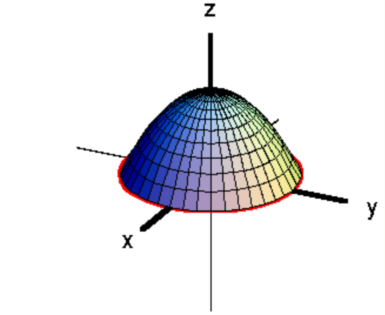
Consider a surface S given by $z = f(x, y)$ over a region R in xy-plane. Suppose that $f(x, y) \geqslant 0$ throughout R and that f has continuous first partial derivatives in R. Note S denotes the portion of the graph of f whose projection in xy-plane. Now, the area of the surface S given by $z = f(x, y)$ over R, where R is a closed region in xy-plane is given by

$$\text{Surface Area} = \iint_R \mathrm{d}S = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\mathrm{d}A$$

Surface Area



(a)



(b)

Surface Area

Note: This formula may also be used if $f(x,y) < 0$ on R.

Examples

(1) Find the Surface area of the paraboloid given by $z = 4 - x^2 - y^2$ for $z \geqslant 0$.

(2) A billowing sail is described as the portion of the graph of $z = 3x + y^2$ that lies over the triangular region R in the xy-plane with vertices $(0,0,0), (0,1,0)$ and $(1,1,0)$. Find the surface area A of the sail.

(3) Find the Surface area S is the part of the paraboloid $z = x^2 + y^2$ cut off by the plane $z = 1$.